

# Explicit connection between conformal field theory and 2+1 Chern-Simons theory

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## Abstract

We give explicit field theoretical representations for the observables of 2+1 dimensional Chern-Simons theory in terms of gauge invariant composites of 2D WZW fields. To test our identification we compute some basic Wilson loop correlators reobtaining known results.

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Since the pionnering work of Witten [1], where it was shown that the Hilbert space of pure Chern Simons (CS) theories is isomorphic to the space of conformal blocks of an underlying Conformal Field Theory (CFT), the connection between 1+1 conformal field theory and 2+1 CS theory has been extensively studied [2, 3, 4, 5, 6].

However, the relations between 1+1 CFT and CS related 2+1 phenomena seem to exceed the original setting of pure CS topological field theory. An still intriguing manifestation of this relation appears in the description of planar condensed matter systems, such as Quantum Hall Effect (QHE) systems [7], where the CS action has found fruitful application. In fact, QHE ground state wave functions have been constructed using CFT data: in refs. [8, 9, 10], wave functions were constructed in terms of the conformal blocks of some CFT's such as the critical Ising model and  $SU(N)_k$  Wess-Zumino-Witten models.

In view of this, it is worthwhile to analyze the connection from different approaches and trade the purely formal point of view of previous works for a more explicit and computationally workable approach. In particular, an explicit identification of the observables in CS theory (Wilson loop operators) in terms of two dimensional conformal fields would be useful to extend the connection to more general cases than the pure CS one. This identification between the observables at the operator level has been only partially addressed in the literature [6].

The purpose of the present paper is to study this point, and to give a direct way to find the explicit expression of an arbitrary Wilson loop operator in terms of the corresponding two dimensional conformal primary operators.

To this end we recall a recent version of the CS – conformal field theory connection [3]. This approach uses the so-called transverse lattice construction [11] in 2+1 dimensions, linking 2-dimensional group  $G$  level  $k$  Wess-Zumino-Witten (WZW) models in each site through left-right asymmetric coupling with  $G$  gauge fields. In fact, it was shown in ref.[3] that the continuum limit of this construction leads to a group  $G$  pure CS theory. Within this approach, Wilson loop correlators were evaluated in the transverse lattice by using representation theory of chiral algebras in each WZW layer. Notice that one distinctive feature of this approach is that it does not undergo a dimensional reduction from the CS manifold to a 1+1 dimensional manifold, but rather deals with a 2+1 dimensional array of CFTs.

Following this route we perform in this note the path integral quantiza-

tion of the theory, giving explicit operator realizations of the Wilson loop operators of the CS theory in terms of gauge invariant composites of two dimensional WZW fields. We show how to evaluate both the partition function and correlation functions for the observables by using decoupling techniques in the path integral framework. We finally test our construction with the computation of some basic Wilson loop correlators; our results are of course in agreement with those originally presented by Witten [1].

For the sake of clarity we first review the transverse lattice construction in ref.[3] and then present the operator realization of the Wilson loop operators in this approach.

The transverse lattice geometry consists of a 2D manifold, which is taken as a Minkowski space  $M_2$ , and a transverse discrete dimension, which is taken as a periodic chain of  $N$  sites ( $S^1$  topology). One introduces a field<sup>1</sup>  $g_n(x^+, x^-)$  on each site, governed by a level  $k$  WZW action [12]

$$kW[g_n] = \frac{k}{8\pi} \int_{M_2} d^2x \text{Tr}(\partial_\mu g_n \partial^\mu g_n^{-1}) + k\Gamma[g_n] \quad (1)$$

where  $g_n$  takes values in a simple Lie group  $G$  and  $\Gamma[g_n]$  is the Wess-Zumino term

$$\Gamma[g] = \frac{1}{12\pi} \int_Y d^3y \epsilon^{ijk} \text{Tr}(g^{-1} \partial_i g g^{-1} \partial_j g g^{-1} \partial_k g) \quad (2)$$

with  $\partial Y = M_2$ . The coupling between two-dimensional layers is accomplished through gauge fields  $A_{\mu,n}$ ;  $g_n$  is left-coupled to  $A_{\mu,n}$  and right-coupled to  $A_{\mu,n+1}$ . The corresponding interaction term in the action is given by [13]

$$\begin{aligned} I[g_n, A_{\pm,n}, A_{\pm,n+1}] &= \frac{k}{2\pi} \int_{M_2} d^2x \text{Tr}[A_{-,n+1} g_n^{-1} \partial_+ g_n - A_{+,n} \partial_- g_n g_n^{-1} \\ &\quad + A_{+,n} g_n A_{-,n+1} g_n^{-1} - \frac{1}{2}(A_{-,n} A_{+,n} + A_{+,n+1} A_{-,n+1})]. \end{aligned} \quad (3)$$

The action  $S_n = kW[g_n] + I[g_n, A_{\pm,n}, A_{\pm,n+1}]$  is not invariant under gauge transformations

$$\begin{aligned} \delta g_n(x^+, x^-) &= \Omega_n(x^+, x^-) g_n(x^+, x^-) - g_n(x^+, x^-) \Omega_{n+1}(x^+, x^-), \\ \delta A_{\pm,n} &= \partial_\pm \Omega_n + [\Omega_n, A_{\pm,n}], \\ \delta A_{\pm,n+1} &= \partial_\pm \Omega_{n+1} + [\Omega_{n+1}, A_{\pm,n+1}]. \end{aligned} \quad (4)$$

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<sup>1</sup>Our conventions for light cone coordinates are  $x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1)$ .

Indeed, the change in the action  $S_n$  reads

$$\delta S_n = \frac{k}{2\pi} \int_{M_2} d^2x \text{Tr} [\Omega_n \epsilon^{\mu\nu} \partial_\mu A_{\nu,n} - \Omega_{n+1} \epsilon^{\mu\nu} \partial_\mu A_{\nu,n+1}] \quad (5)$$

which is related to the non-Abelian anomaly in two dimensions [14]. Note that in case the coupling was left-right symmetric ( $A_{\pm,n} = A_{\pm,n+1}$ ) the variation would vanish; this would be equivalent to gauging the anomaly free vector subgroup of the left and right global quiral symmetry of the WZW model.

The entire system consists of a periodic chain of 2D layers. Its action is simply given by

$$S = \sum_{n=1}^N S_n. \quad (6)$$

This system is gauge invariant because of the cancellation of the second term in (5) with the first term coming from the variation corresponding to the following site. We will refer to this interplay as gauge invariance of the links (Fig.1).

In order to make contact with the 2+1 CS theory, following ref.[3], we represent the link field  $g_n$  as a function on the transverse lattice,

$$g_n = \exp(-\int_{x^3}^{x^3+a} A_3(x^+, x^-, x^3) dx^3). \quad (7)$$

Here  $a$  is the spacing of the lattice and  $x^3 = na$  will become a continuous coordinate as  $a \rightarrow 0$ ,  $N \rightarrow \infty$ , while  $Na = L$  remains constant. In this limit the phase exponent in (7) can be written as  $-aA_3$ , with  $A_3$  evaluated in  $x^3 = (n + 1/2)a$ . Using this parametrization in eq.(6) one readily obtains

$$\lim_{\substack{a \rightarrow 0 \\ N \rightarrow \infty}} S = \frac{k}{2\pi} \int_{M_2 \times S^1} d^3x \epsilon^{ijk} \text{Tr} [A_i \partial_j A_k - \frac{2}{3} A_i A_j A_k] \equiv S_{CS}, \quad (8)$$

which corresponds to the level  $k$  CS action for the gauge group  $G$ . It should be stressed that the same result is obtained starting from the off critical WZW action instead of action (1).

Once the connection (8) between classical actions is established, one is naturally led to study the quantization of both theories and the relation

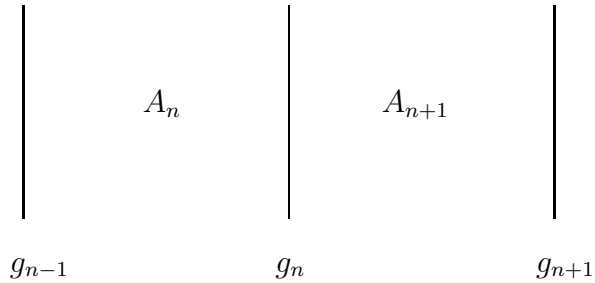


Figure 1: Gauge invariance of  $S$  is achieved by compensation of gauge anomalies at each link between WZW layers.

between their observables. This study has been performed in ref.[3] in the framework of canonical quantization, using the representation theory of the Virasoro algebra for the 1+1-dimensional layers and then solving the constraints arising from gauge invariance of  $S$  in eq.(6). In this way, the correspondence between the physical states of the lattice CS theory and some Wilson loops of the continuum CS theory has been proved. In our investigation, we shall instead work in the path-integral approach. This will allow us to reach our aim, that is to construct explicit operator expressions for Wilson loops in the lattice CS theory.

We start from the partition function of the lattice CS theory

$$\mathcal{Z} = \int \prod_{n=1}^N \mathcal{D}A_{\mu,n} \mathcal{D}g_n e^{iS} \quad (9)$$

with  $S$  given by eq.(6).

Before presenting our approach, we have to mention that the path integral (9) can be effectively reduced to a 2-dimensional theory<sup>2</sup>. In fact, the results presented by Witten in ref.[4] show that one could shrink the transverse dimension to a point instead of taking the continuum limit (which is not

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<sup>2</sup>We thank G. Thompson for pointing this out to us

surprising since being the pure CS theory a topological field theory, the length of the transverse dimension should be irrelevant). To be more precise, Witten has shown that

$$\begin{aligned} \int \mathcal{D}A_{\mu,n+1} \mathcal{D}g_n \mathcal{D}g_{n+1} e^{i(kW[g_n] + I[g_n, A_n, A_{n+1}] + kW[g_{n+1}] + I[g_{n+1}, A_{n+1}, A_{n+2}])} = \\ = \int \mathcal{D}g e^{i(kW[g] + I[g, A_n, A_{n+2}])}. \end{aligned} \quad (10)$$

This reduction formula eliminates the degrees of freedom associated to the site  $n + 1$ , and its repeated application finally leads to the consideration of the 2-dimensional left-right symmetric  $G/G$  coset model

$$\int \mathcal{D}A_\mu \mathcal{D}g e^{i(kW[g] + I[g, A, A])}. \quad (11)$$

This model has been studied in the literature (see for instance [5, 6]). In particular, in [6] the authors reobtain, as we will do, the already known results for Wilson loop expectation values in CS topological field theory [1]. However, our aim is to give a way to construct operator realizations for Wilson loops in terms of conformal fields, and for this purpose the present approach is more appropriate, as we will show.

We then maintain every site in the chain and perform the following (decoupling [13, 15]) change of variables

$$\begin{aligned} A_{+,n} &= f_n^{-1} \partial_+ f_n, \\ A_{-,n} &= h_n^{-1} \partial_- h_n, \\ g_n &= f_n^{-1} g_n^{(0)} h_{n+1}. \end{aligned} \quad (12)$$

After this change the action  $S$  takes the form

$$S = \sum_{n=1}^N (kW[g_n^{(0)}] - kW[f_n h_n^{-1}]), \quad (13)$$

where the Polyakov-Wiegmann identity

$$W[gh] = W[g] + W[h] - \frac{1}{2\pi} \int d^2x \text{Tr}(g^{-1} \partial_+ g \partial_- h h^{-1}) \quad (14)$$

has been used [16].

The Jacobian associated with transformation (12) is given by

$$\mathcal{D}A_{+,n}\mathcal{D}A_{-,n} = \det(D_{+,n}^{Adj})\det(D_{-,n}^{Adj})\mathcal{D}f_n\mathcal{D}h_n \quad (15)$$

where  $D_{\pm,n}^{Adj} = \partial_{\pm} + [A_{\pm,n}, \ ]$ . The adjoint determinants can be written as

$$\det(D_{+,n}^{Adj})\det(D_{-,n}^{Adj}) = \mathcal{Z}_{gh,n}e^{-2iC_v W[f_n h_n^{-1}]} \quad (16)$$

where  $C_v$  is the adjoint Casimir of  $G$ . Here we have adopted a regularization prescription that preserves the gauge invariance (or anomaly cancellation) associated to each link. The explicit form of the ghost partition function in eq.(16) will not be relevant in the following.

As the last step to arrive at the decoupled partition function we change  $f_n \rightarrow \tilde{f}_n = f_n h_n^{-1}$  (with unit Jacobian). Then the integral over  $h_n$  factorizes as  $Vol\hat{G}$ . This procedure tantamounts to fixing the  $A_- = 0$  gauge. The decoupled partition function now reads

$$\mathcal{Z} = \mathcal{Z}_{gh} \int \prod_{n=1}^N \mathcal{D}g_n^{(0)} e^{ikW[g_n^{(0)}]} \int \prod_{n=1}^N \mathcal{D}\tilde{f}_n e^{-i(2C_v+k)W[\tilde{f}_n]}. \quad (17)$$

Thus the partition function factorizes in  $3N$  sectors, corresponding to  $N$  free ghost systems,  $N$   $G$ -valued WZW fields  $g_n$  with level  $k$  and  $N$   $G$ -valued WZW fields  $\tilde{f}_n$  with negative level  $-(k + 2C_v)$ . For each site  $n$  one has a conformal field theory with vanishing total central charge, built up from the different sectors: a ghost sector contributing with  $c_{gh} = -2dimG$ , a level  $k$  WZW sector contributing with  $c_k = k dimG/(k + C_v)$  and a negative level WZW sector contributing with  $c_{-(k+2C_v)} = (k + 2C_v) dimG/(k + C_v)$ . The level  $k$  field  $g_n^{(0)}$  can be thought as localized in the  $n$ -th site and the level  $-(k + 2C_v)$  field  $\tilde{f}_n$  as localized in the link between the  $(n-1)$ -th and the  $n$ -th sites. Notice that, although the partition function of the theory is completely decoupled, BRST quantization condition connects the different sectors (as will be apparent in eq.(21)) in order to ensure unitarity [15, 17].

Once we have written the partition function in the simple form (17), we turn our attention to the construction of the observables of this theory. We will then show that the lattice version observables naturally lead to Wilson loop operators in the continuum CS limit.

The observables in the lattice are constructed from gauge invariant composites of WZW fields, which in turn belong to integrable representations

of the group  $G$ . In order to keep a simple notation we will discuss the specific example of  $SU(2)_k$ , although the extension to  $SU(N)_k$  and more general groups is straightforward. The integrable representations of  $SU(2)_k$  are characterized by the spin  $j = 0, 1/2, \dots, k/2$  [18]. The basic field  $g_n$  in eq.(1) is taken in the fundamental representation of  $SU(2)$  ( $j = 1/2$ ). Fields  $g_n^{(j)}$  in higher spin representations are constructed as appropriately symmetrized direct products of fields in the fundamental one. As gauge invariance of the lattice model is achieved in each link, in order to construct gauge invariant composites one has to take the trace of the product of  $g_n$  fields in each site, all of them in the same representation. That is,

$$R_j(x^+, x^-) = Tr_j \prod_{n=1}^N g_n^{(j)}(x^+, x^-) \quad (18)$$

where  $Tr_j$  means matrix trace in the representation of spin  $j$ .

To see the connection of these fields with Wilson loop operators one has to use eq.(7) (now in the representation  $j$ ) obtaining

$$R_j(x^+, x^-) = Tr_j \prod_{n=1}^N e^{-aA_3(x^+, x^-, na+a/2)} \quad (19)$$

which in the continuum limit gives

$$R_j(x^+, x^-) \longrightarrow_{a \rightarrow 0} Tr_j P(e^{-\int_C dx^\mu A_\mu}). \quad (20)$$

This is the expression for Wilson loop operators winding once around a circle  $C$  passing through  $(x^+, x^-)$  in each layer and carrying flux in the representation  $j$ .

While the limit taken in (20) is a standard procedure in lattice gauge theories, our main result is that the identification is also valid at the level of quantum operators; in other words, one obtains the same result by taking the continuum limit before computing quantum expectation values, thus working in the CS theory, or after computing quantum expectation values in 2-dimensional conformal field theory.

We will check the previous statement by evaluating up to three point correlators using the decoupled picture (17). In this picture the fields in eq.(18) can be written in terms of the decoupled variables  $g_n^{(0)}$ ,  $\tilde{f}_n$  as

$$R_j(x^+, x^-) = Tr_j \prod_{n=1}^N (\tilde{f}_n^{(j)})^{-1}(x^+, x^-) g_n^{(0)(j)}(x^+, x^-). \quad (21)$$



Thus, correlators involving  $R_j$ 's factorize in the level  $k$  and level  $-(k+4)$  WZW sectors.

The conformal dimensions of the primary fields in a level  $K$   $SU(2)$  WZW theory are given by  $C_j/(K+C_v)$ , where  $C_j = j(j+1)$  is the Casimir in the spin  $j$  representation and the adjoint Casimir is  $C_v = 2$ . For the fields  $g_n^{(0)(j)}$  and  $\tilde{f}_n^{(j)}$  we have

$$\begin{aligned} h[g_n^{(0)(j)}] &= \frac{j(j+1)}{k+2} \\ h[\tilde{f}_n^{(j)}] &= -\frac{j(j+1)}{k+2} \end{aligned} \quad (22)$$

and hence the conformal dimension of  $R_j$  vanishes. This implies that the correlators are independent of the coordinates<sup>3</sup>  $(x^+, x^-)$ , this being in correspondence with the topological nature of Wilson loop operators in the pure CS theory.

The one point correlator vanishes except for the trivial  $j = 0$  representation, in which the fields correspond to the identity operator, giving

$$\langle R_j \rangle = \delta_{j0}. \quad (23)$$

For the two point correlator one has to evaluate

$$\langle R_{j_1}(x_1) R_{j_2}(x_2) \rangle = \langle \text{Tr}_{j_1} \left( \prod_{n=1}^N g_n^{(j_1)}(x_1) \right) \text{Tr}_{j_2} \left( \prod_{n=1}^N g_n^{(j_2)}(x_2) \right) \rangle. \quad (24)$$

In order to give a detailed derivation we introduce indices for each representation  $j$ . As  $g^{(j)}$  transforms in the representation  $j \times j$  of  $SU(2)_L \times SU(2)_R$ , we write  $g_{\alpha\alpha'}^{(j)}$  with  $\alpha(\alpha') = -j, \dots, j$  corresponding to the left (right) representation. One can then factorize eq.(24) as

$$\begin{aligned} & \prod_{n=1}^N \langle (g_n^{(0)(j_1)})_{\alpha_n \alpha'_n}(x_1) (g_n^{(0)(j_2)})_{\beta_n \beta'_n}(x_2) \rangle \\ & \prod_{n=1}^N \langle (\tilde{f}_n^{(j_1)})_{\alpha'_{n-1} \alpha_n}^{-1}(x_1) (\tilde{f}_n^{(j_2)})_{\beta'_{n-1} \beta_n}^{-1}(x_2) \rangle \end{aligned} \quad (25)$$

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<sup>3</sup>This fact is a direct consequence of conformal symmetry for the two and three point correlators. In the four point case the analysis is more involved but coordinate independence can be proved following ref.[19] (see section 3.2).

where each v.e.v. in the first line of (29) is evaluated in the theory with action  $kW[g_n^{(0)}]$  and each one in the second line is evaluated using the action  $-(k+4)W[f_n]$ .

It is known that the two point correlator of primary fields in a conformal field theory is completely determined by conformal symmetry. For an  $SU(2)$  WZW theory it reads [21]

$$\begin{aligned} \langle (g^{(j_1)})_{\alpha\alpha'}(x_1)(g^{(j_2)})_{\beta\beta'}(x_2) \rangle = \\ (-1)^{2j_1-\alpha-\beta}\delta_{j_1j_2}\delta_{\alpha,-\beta}\delta_{\alpha',-\beta'}(2x_{12}^+x_{12}^-)^{-2h_{j_1}}, \end{aligned} \quad (26)$$

where  $x_{12} = x_1 - x_2$  and  $h_j$  is the conformal dimension of the fields in the representation  $j$ . Using this result in eq.(25) one readily obtains

$$\langle R_{j_1}(x_1)R_{j_2}(x_2) \rangle = \delta_{j_1j_2} \quad (27)$$

For the three point correlator the derivation is analogous. One has to evaluate

$$\begin{aligned} \langle R_{j_1}(x_1)R_{j_2}(x_2)R_{j_3}(x_3) \rangle = \\ \langle Tr_{j_1}(\prod_{n=1}^N g_n^{(j_1)}(x_1))Tr_{j_2}(\prod_{n=1}^N g_n^{(j_2)}(x_2))Tr_{j_3}(\prod_{n=1}^N g_n^{(j_3)}(x_3)) \rangle, \end{aligned} \quad (28)$$

which factorizes as

$$\begin{aligned} \prod_{n=1}^N \langle (g_n^{(0)(j_1)})_{\alpha_n\alpha'_n}(x_1)(g_n^{(0)(j_2)})_{\beta_n\beta'_n}(x_2)(g_n^{(0)(j_3)})_{\gamma_n\gamma'_n}(x_3) \rangle \\ \prod_{n=1}^N \langle (\tilde{f}_n^{(j_1)})_{\alpha'_{n-1}\alpha_n}^{-1}(x_1)(\tilde{f}_n^{(j_2)})_{\beta'_{n-1}\beta_n}^{-1}(x_2)(\tilde{f}_n^{(j_3)})_{\gamma'_{n-1}\gamma_n}^{-1}(x_3) \rangle. \end{aligned} \quad (29)$$

The three point correlator of primary fields in a level  $K$  conformal field theory is determined by conformal symmetry only up to some coefficients which can be evaluated from operator product expansion,

$$\begin{aligned} \langle (g^{(j_1)})_{\alpha\alpha'}(x_1)(g^{(j_2)})_{\beta\beta'}(x_2)(g^{(j_3)})_{\gamma\gamma'}(x_3) \rangle = C \begin{pmatrix} j_1 & \alpha & \alpha' \\ j_2 & \beta & \beta' \\ j_3 & \gamma & \gamma' \end{pmatrix}_K \times \\ (2x_{12}^+x_{12}^-)^{h_{j_3}-h_{j_1}-h_{j_2}}(2x_{23}^+x_{23}^-)^{h_{j_1}-h_{j_2}-h_{j_3}}(2x_{13}^+x_{13}^-)^{h_{j_2}-h_{j_1}-h_{j_3}}, \end{aligned} \quad (30)$$

where  $x_{ij} = x_i - x_j$ . For a level  $K$   $SU(2)$  WZW theory the coefficients  $C$  have been given in ref.[21] in the form

$$C \begin{pmatrix} j_1 & \alpha & \alpha' \\ j_2 & \beta & \beta' \\ j_3 & \gamma & \gamma' \end{pmatrix}_K = \begin{bmatrix} j_1 & j_2 & j_3 \\ \alpha & \beta & \gamma \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j_3 \\ \alpha' & \beta' & \gamma' \end{bmatrix} \rho_K(j_1, j_2, j_3) \quad (31)$$

for  $|j_1 - j_2| \leq j_3 \leq \min(j_1 + j_2, k - j_1 - j_2)$  and zero otherwise. The first two factors in eq.(31) are the Wigner  $3j$ -symbols and the function  $\rho_K(j_1, j_2, j_3)$  can be written in terms of  $\Gamma$  functions; for our purposes it will be important that

$$\rho_k(j_1, j_2, j_3) \rho_{-(k+4)}(j_1, j_2, j_3) = 1. \quad (32)$$

The exponents in eq.(30) are given in terms of the conformal weights of the primary fields; they cancel in our case in virtue of eq.(22).

Using the results above the correlator in eq.(28) can be written as

$$\begin{aligned} \langle R_{j_1}(x_1) R_{j_2}(x_2) R_{j_3}(x_3) \rangle &= \prod_{n=1}^N C \begin{pmatrix} j_1 & \alpha_n & \alpha'_n \\ j_2 & \beta_n & \beta'_n \\ j_3 & \gamma_n & \gamma'_n \end{pmatrix}_k C \begin{pmatrix} j_1 & \alpha'_{n-1} & \alpha_n \\ j_2 & \beta'_{n-1} & \beta_n \\ j_3 & \gamma'_{n-1} & \gamma_n \end{pmatrix}_{-(k+4)} \\ &= \left( \begin{bmatrix} j_1 & j_2 & j_3 \\ \alpha_n & \beta_n & \gamma_n \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j_3 \\ \alpha_n & \beta_n & \gamma_n \end{bmatrix} \right)^N \left( \begin{bmatrix} j_1 & j_2 & j_3 \\ \alpha'_i & \beta'_i & \gamma'_i \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j_3 \\ \alpha'_i & \beta'_i & \gamma'_i \end{bmatrix} \right)^N \end{aligned} \quad (33)$$

Orthogonality relations between the  $3j$ -symbols allows one to express the result as

$$\langle R_{j_1}(x_1) R_{j_2}(x_2) R_{j_3}(x_3) \rangle = \delta(j_1, j_2, j_3) \quad (34)$$

where  $\delta(j_1, j_2, j_3)$  means 1 in case that  $j_1$ ,  $j_2$  and  $j_3$  satisfy a triangular condition and 0 otherwise.

Our results (23), (27) and (34) coincide with the expectation value of one, two and three unknotted Wilson loops given by Witten [1] in terms of the fusion rules of the symmetry group, which for  $SU(2)_k$  explicitly read [22]

$$N_{j_1 j_2}^0 = \delta_{j_1 j_2} \quad (35)$$

$$N_{j_1 j_2 j_3} = \begin{cases} 1 & \text{if } |j_1 - j_2| \leq j_3 \leq \min(j_1 + j_2, k - j_1 - j_2) \\ 0 & \text{otherwise.} \end{cases} \quad (36)$$

Let us recall that these correlators in the manifold  $S^2 \times S^1$  are the basic results that, through the use of surgery techniques, allow for the computation of the link invariants associated to arbitrary links in  $S^3$  [1].

In summary, we have given an explicit operator realization of the Wilson loop operators (winding once around  $S^1$  and carrying flux in the representation  $j$ ) in terms of gauge invariant products of WZW fields in the transverse lattice formulation. We have also shown how to decouple the lattice partition function and in order to test this scheme we have explicitly evaluated correlators of up to three unknotted Wilson loops. This results are not new, but they are of course in agreement with those found in the literature [1, 6]. However, our Lagrangian approach is new and complimentary to the direct 3-dimensional CS approach and the 2-dimensional Hamiltonian approach, and could be interesting in regard to generalizations. Following our presentation, one could for instance look for explicit representations of Wilson loop operators in less trivial topologies.

As stated in the introduction, the explicit identification between observables in the CS theory and two dimensional primary fields could be useful to exploit the connection in more general cases. In particular, it is interesting to analyze perturbations that violate the topological symmetry of the CS theory but preserve the integrability of the conformal theories in the layers of the transverse lattice construction. Another interesting open route arises from the fact that our treatment of the underlying conformal symmetry does not use representation theory of chiral algebras but rather represents observables explicitly. It makes possible to find connections with an alternative description of conformal field theory, namely the spinon formulation [23],[24]. These issues are under current investigation.

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